

# On 2-dimensional Kähler metrics with one holomorphic isometry<sup>1</sup>

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## Abstract

We show how to write any Kähler metric of complex dimension 2 admitting a holomorphic isometry as a simple 1-real-function deformation of a Gibbons-Hawking metric. Hyper-Kähler metrics with a tri-holomorphic isometry (Gibbons-Hawking metrics) or with a mono-holomorphic isometry are recovered for particular values of the additional function. The new general metric can be used as an Ansatz in several interesting physical problems.

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<sup>1</sup>**Important notice:** After the first submission of this paper to the arXiv, we have realized that the main result presented in it was implicitly contained in Ref. [10], as explained in Section 2 of Ref. [15], something neither we nor the experts we consulted before the submission were aware of. While this paper cannot be published in a regular scientific journal, we think it still can be useful for the scientific community.

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# Introduction

Kähler spaces of complex dimension 2 play very important rôles in physics. Of particular interest for us is their occurrence as *base spaces* for supersymmetric solutions of minimal Fayet-Iliopoulos (FI) U(1)–gauged supergravity in 5 and 6 dimensions (see Refs. [1, 2, 3, 4]). In order to give a closed form to these solutions, a closed form for all 2-dimensional Kähler metrics would be needed. This is possible if one writes them as second derivatives of the Kähler potential, but then, the differential equations that determine the rest of the solutions' fields would be two orders higher and much more difficult to solve.

In the ungauged case one faces a similar problem: finding a closed form for all the hyper-Kähler metric of real dimension 4, and a partial, yet extremely good, solution is to consider those that admit a triholomorphic isometry. All these metrics can be written in a simple closed form in terms of a single real function traditionally denoted by  $H$ , and are known as Gibbons-Hawking (GH) metrics [5, 6]. Furthermore, these metrics can be dimensionally reduced along the isometric direction, establishing fruitful relations between 5- and 4-dimensional supersymmetric supergravity solutions. Supersymmetric solutions of 5-dimensional supergravity with a GH base include (single or multicenter, static and rotating) black-holes and black rings.

It is, then, natural, to consider Kähler metrics admitting one holomorphic isometry in the U(1) gauged case, but no closed form for them has been given in the literature. The goal of this paper is to close this gap: we are going to show how any Kähler metric of complex dimension 2 admitting one holomorphic Killing vector can be written in a simple way in terms of two independent real functions  $H, W$ . Particular cases such as GH metrics, or hyperKähler metrics with mono-holomorphic isometries [7, 8, 9] or the scalar-flat Kähler metrics with a holomorphic isometry considered by Lebrun in Ref. [10] are contained in this general form and can be recovered by imposing additional conditions on the function  $W$ .

## 1 4-d Kähler metrics with one holomorphic isometry

**Theorem:** Any Kähler metric of real dimension 4 admitting a holomorphic isometry can be locally written in the form

$$ds^2 = H^{-1} (dz + \chi)^2 + H \left\{ (dx^2)^2 + W^2(\vec{x})[(dx^1)^2 + (dx^3)^2] \right\}, \quad (1.1)$$

with the functions  $H$  and  $W$ , and the 1-form  $\chi$ , depending only on the three coordinates  $x^i, i = 1, 2, 3$ , and satisfying the constraints

$$\begin{aligned}
(d\chi)_{\underline{1}\underline{2}} &= \partial_{\underline{3}}H, \\
(d\chi)_{\underline{2}\underline{3}} &= \partial_{\underline{1}}H, \\
(d\chi)_{\underline{3}\underline{1}} &= \partial_{\underline{2}}(W^2H).
\end{aligned} \tag{1.2}$$

Conversely, any metric of the above form is a Kähler metric admitting a holomorphic isometry.

**Remark:** The integrability condition of the above three equations is

$$\mathfrak{D}^2H \equiv \partial_{\underline{1}}\partial_{\underline{1}}H + \partial_{\underline{2}}\partial_{\underline{2}}(W^2H) + \partial_{\underline{3}}\partial_{\underline{3}}H = 0. \tag{1.3}$$

Notice that, in general, this equation is not (proportional to) the Laplace equation in the 3-dimensional metric. The 3-dimensional Laplacian takes the form

$$\overline{\nabla}^2H = \frac{1}{W^2} \left[ \partial_{\underline{1}}\partial_{\underline{1}}H + \partial_{\underline{2}}(W^2\partial_{\underline{2}}H) + \partial_{\underline{3}}\partial_{\underline{3}}H \right], \tag{1.4}$$

and, therefore, the integrability equation is proportional to the Laplace equation for  $x^2$ -independent conformal factors  $W$ .

On the other hand, locally, the metric (1.1) is entirely determined by the two real functions  $H$  and  $W$ . Once a solution  $(H, W)$  of Eq. (1.3) has been found, the 1-form  $\chi$  is determined from (1.2) up to an irrelevant closed 1-form.

**Proof of the theorem:** Any 4-dimensional Euclidean metric admitting one isometry can be written in the form

$$d\hat{s}^2 = H^{-1}(dz + \chi)^2 + H\gamma_{\underline{ij}}dx^{\underline{i}}dx^{\underline{j}}, \tag{1.5}$$

where  $z = x^{\sharp}$  is the coordinate adapted to the isometry and where the 3-dimensional function  $H$ , the 1-form  $\chi = \chi_{\underline{i}}dx^{\underline{i}}$  and the metric  $\gamma_{\underline{ij}}dx^{\underline{i}}dx^{\underline{j}}$ ,  $i, j = 1, 2, 3$  are  $z$ -independent and orthogonal to the Killing vector  $k^{\underline{m}} = \delta_z^{\underline{m}}$ . We denote the coordinate base indices by  $\{\underline{m}\} = \{z, \underline{i}\}$  and the tangent space indices by  $\{m\} = \{\sharp, \underline{i}\}$ . We will denote 3-dimensional structures (connection, curvature etc.) by an overline.

A convenient basis of Vierbeins is

$$\begin{cases} \hat{V}^{\sharp} &= H^{-1/2}(dz + \chi), \\ \hat{V}^{\underline{i}} &= H^{1/2}v^{\underline{i}}, \end{cases} \quad \begin{cases} \hat{V}_{\sharp} &= H^{1/2}\partial_z, \\ \hat{V}_{\underline{i}} &= H^{-1/2}(\partial_{\underline{i}} - \chi_{\underline{i}}\partial_z), \end{cases} \tag{1.6}$$

where  $v^{\underline{i}} = v^{\underline{i}}_{\underline{j}}dx^{\underline{j}}$  are Dreibeins of the metric  $\gamma_{\underline{ij}}$ ,  $\partial_{\underline{i}} \equiv v^{\underline{j}}_{\underline{i}}\partial_{\underline{j}}$  and  $\chi_{\underline{i}} \equiv v^{\underline{j}}_{\underline{i}}\chi_{\underline{j}}$ .

The non-vanishing components of the spin connection 1-form, defined through the structure equation  $\mathcal{D}\hat{V}^m \equiv d\hat{V}^m - \omega^m_n \wedge \hat{V}^n = 0$  are

$$\begin{aligned}
\omega_{\sharp\sharp i} &= \frac{1}{2}H^{-3/2}\partial_i H, & \omega_{\sharp ij} &= \frac{1}{2}H^{-3/2}(d\chi)_{ij}, \\
\omega_{i\sharp j} &= \omega_{\sharp ij}, & \omega_{kij} &= H^{-1/2}\bar{\omega}_{kij} + H^{-3/2}\partial_{[i}H\delta_{j]k},
\end{aligned} \tag{1.7}$$

where  $(d\chi)_{ij} = 2v_i^k v_j^l \partial_{[j}\chi_{l]}$  and  $\bar{\omega}_{kij}$  is the 3-dimensional connection defined by  $\bar{\mathcal{D}}v^i = dv^i - \bar{\omega}^i_j \wedge v^j = 0$ .

For the manifold to be Kähler, there must exist a globally defined almost complex structure  $J_{\underline{mn}}$ ,

$$J_{\underline{p}}^m J_{\underline{n}}^p = -\delta_{\underline{n}}^m, \tag{1.8}$$

with respect to which the metric  $h_{\underline{mn}}$  is Hermitian,

$$h_{\underline{mn}} J_{\underline{p}}^m J_{\underline{q}}^n = h_{\underline{pq}}, \tag{1.9}$$

and which is covariantly constant with respect to the Levi-Civita connection,

$$\nabla_{\underline{m}} J_{\underline{p}}^n = 0. \tag{1.10}$$

Eqs. (1.8) and (1.9) imply that  $J_{\underline{mn}} \equiv h_{\underline{mp}} J_{\underline{n}}^p$  is antisymmetric: *i.e.* it is a 2-form known as the Kähler 2-form. It is obvious from the covariant constancy of  $J$  that the Kähler 2-form is closed. Assuming the other two conditions are met, the closedness of the Kähler 2-form, its covariant constancy or that of the complex structure with respect to the Levi-Civita connection are equivalent. These statements are also equivalent to the statement that the holonomy of  $h_{\underline{mn}}$  is contained in  $U(2)$  and  $J$  is the associated  $U(2)$ -structure [11].

In flat four-dimensional indices these conditions are equivalent to

$$J_{mn} J_{np} = -\delta_{mn}, \tag{1.11}$$

$$J_{mn} = -J_{nm}, \tag{1.12}$$

$$\nabla_m J_{np} = 0. \tag{1.13}$$

A  $J$  that satisfies the first two conditions and can always be chosen is given by

$$(J_{mn}) \equiv \begin{pmatrix} 0_{2 \times 2} & \mathbb{1}_{2 \times 2} \\ -\mathbb{1}_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}. \tag{1.14}$$

We have chosen it to be antiselfdual for the sake of convenience.

Since in this form  $J$  is constant, the third condition Eq. (1.13) is equivalent to the vanishing of the commutator of  $J$  with all the components  $m$  of the spin connection 1-form  $\omega_m^n{}_p$

$$[\omega_m, J] = 0. \quad (1.15)$$

Using the explicit form of the components of the spin connection in Eqs. (1.7), the  $m = \sharp$  component of this equation gives

$$(d\chi)_{12} = \partial_3 H, \quad (1.16)$$

$$(d\chi)_{23} = \partial_1 H, \quad (1.17)$$

while the  $m = i$  components impose the following conditions on the components of the spin connection of the 3-dimensional metric  $\gamma_{ij}$ :

$$\bar{\omega}_{221} = \bar{\omega}_{223} = \bar{\omega}_{321} = \bar{\omega}_{123} = 0, \quad (1.18)$$

$$\bar{\omega}_{112} = \bar{\omega}_{332} = \frac{1}{2H} [(d\chi)_{13} + \partial_2 H]. \quad (1.19)$$

(Observe that the components  $\bar{\omega}_{113}$ ,  $\bar{\omega}_{213}$  and  $\bar{\omega}_{313}$  are not constrained by the Kähler condition.)

The last condition that we have to impose on the metric Eq. (1.5) is that the isometry is holomorphic, that is: the Killing vector  $k$  preserves the complex structure

$$\mathcal{L}_k J = 0. \quad (1.20)$$

However, given the choices made here, this turns out to be automatically true and does not provide any further conditions.

We now remind the reader that the condition  $\mathcal{L}_k J = 0$  together with the closedness of the Kähler 2-form lead to

$$\mathcal{L}_k J = i_k(dJ) + d(i_k J) = d(i_k J) = 0, \quad (1.21)$$

which implies the existence of a real function  $\mathcal{P}$  known as the *momentum map* such that

$$i_k J = -d\mathcal{P}. \quad (1.22)$$

The conditions (1.18) imply that  $v^2$  is a closed 1-form,  $dv^2 = 0$ , which means that it is possible to choose a coordinate  $x^2$  such that, locally,

$$v^2 = dx^2. \quad (1.23)$$

This can also be seen in a different way: given the form of  $J$  in (1.14), the Kähler form is

$$\mathcal{J} = \frac{1}{2} J_{mn} e^m \wedge e^n = e^\sharp \wedge e^2 + e^1 \wedge e^3, \quad (1.24)$$

which implies, since  $e^\sharp = H^{1/2} k_{\underline{m}} dx^{\underline{m}}$  and  $e^2 = H^{1/2} v^2$ ,

$$v^2 = \iota_k \mathcal{J}. \quad (1.25)$$

Comparing this equation with Eq. (1.22) we see that

$$v^2 = -d\mathcal{P}, \quad (1.26)$$

and we conclude that we have chosen, as coordinate  $x^2$ , (minus) the momentum map  $x^2 = -\mathcal{P}$ .

Apart from the condition  $dv^2 = 0$ , the information on the 3-dimensional metric given by Eqs. (1.18) and (1.19) can be summarized in the conditions

$$dv^1 \wedge v^1 = dv^3 \wedge v^3, \quad dv^1 \wedge v^3 = -dv^3 \wedge v^1. \quad (1.27)$$

Introducing another two coordinates  $x^{1,3}$ , in general one will have  $v^1 = v^1_{\underline{i}} dx^{\underline{i}}$  and  $v^3 = v^3_{\underline{i}} dx^{\underline{i}}$ , with  $i = 1, 2, 3$  but the components  $v^1_{\underline{2}}$  and  $v^3_{\underline{2}}$  can always be set to zero with a coordinate change  $x^{1,3} \rightarrow F^{1,3}(\vec{x})$  such that

$$\partial_{\underline{2}} F^1 = \frac{v^1_{\underline{3}} v^3_{\underline{2}} - v^1_{\underline{2}} v^3_{\underline{3}}}{v^1_{\underline{1}} v^3_{\underline{3}} - v^3_{\underline{1}} v^1_{\underline{3}}}, \quad \partial_{\underline{2}} F^3 = \frac{v^3_{\underline{1}} v^1_{\underline{2}} - v^3_{\underline{2}} v^1_{\underline{1}}}{v^1_{\underline{1}} v^3_{\underline{3}} - v^3_{\underline{1}} v^1_{\underline{3}}}. \quad (1.28)$$

If  $v^{1,3}_{\underline{2}} = 0$ , then Eqs. (1.27) imply the following relations between the Dreibein components and their partial derivatives with respect to the coordinate  $x^2$  hold:

$$\begin{aligned} \partial_{\underline{2}} v^3_{\underline{1}} &= \frac{\partial_{\underline{2}} v^1_{\underline{1}} (v^1_{\underline{1}} v^1_{\underline{3}} + v^3_{\underline{1}} v^3_{\underline{3}}) - \partial_{\underline{2}} v^1_{\underline{3}} [(v^1_{\underline{1}})^2 + (v^3_{\underline{1}})^2]}{v^1_{\underline{1}} v^3_{\underline{3}} - v^3_{\underline{1}} v^1_{\underline{3}}}, \\ \partial_{\underline{2}} v^3_{\underline{3}} &= \frac{\partial_{\underline{2}} v^1_{\underline{1}} [(v^1_{\underline{1}})^2 + (v^3_{\underline{1}})^2] - \partial_{\underline{2}} v^1_{\underline{3}} (v^1_{\underline{1}} v^1_{\underline{3}} + v^3_{\underline{1}} v^3_{\underline{3}})}{v^1_{\underline{1}} v^3_{\underline{3}} - v^3_{\underline{1}} v^1_{\underline{3}}}. \end{aligned} \quad (1.29)$$

For a fixed value of  $x^2$  (and treating it as a constant) there always exists a coordinate change  $x^{1,3} \rightarrow G^{1,3}(x^1, x^3)$  allowing to rewrite the 2-dimensional metric  $ds_2^2 = (v^1)^2 + (v^3)^2$  in conformally flat form  $ds_2^2 = W^2(\vec{x})[(dx^1)^2 + (dx^3)^2]$ . The derivatives of the functions  $G^{1,3}$  that do the trick satisfy the conditions

$$\partial_{\underline{1}} G^3 = A \partial_{\underline{1}} G^1 + B \partial_{\underline{3}} G^1, \quad \partial_{\underline{3}} G^3 = A \partial_{\underline{3}} G^1 - B \partial_{\underline{1}} G^1, \quad (1.30)$$

with

$$A = -\frac{v^1_{\underline{1}} v^1_{\underline{3}} + v^3_{\underline{1}} v^3_{\underline{3}}}{(v^1_{\underline{3}})^2 + (v^3_{\underline{3}})^2}, \quad B = \pm \frac{v^1_{\underline{3}} v^3_{\underline{1}} - v^1_{\underline{1}} v^3_{\underline{3}}}{(v^1_{\underline{3}})^2 + (v^3_{\underline{3}})^2}. \quad (1.31)$$

In general, if the non-vanishing components of the Dreibein depend on  $x^2$ , the functions  $A$  and  $B$  depend on  $x^2$ , and the functions  $G^{1,3}$  cannot satisfy the above equations being independent of  $x^2$ . On the other hand, if  $G^{1,3}$  depended on  $x^2$  the above equations would not make sense as we would have to include partial derivatives with respect to  $x^2$ .

In the present case, however, it turns out that Eqs. (1.29) imply that  $\partial_2 A = \partial_2 B = 0$ , guaranteeing that the same coordinate change with  $x^2$ -independent  $G^{1,3}$  allows to write the 2-dimensional metric in conformally flat form even if the components of  $v^{1,3}$  depend on the third coordinate  $x^2$ .

We conclude that we can always choose coordinates in the 2-dimensional metric such that the non-trivial Dreibein are given by

$$v^{1,3} = W(\vec{x}) dx^{1,3}, \quad (1.32)$$

and the 3-dimensional metric is

$$d\bar{s}^2 = \gamma_{ij} dx^i dx^j = (dx^2)^2 + W^2(\vec{x})[(dx^1)^2 + (dx^3)^2]. \quad (1.33)$$

Computing explicitly the spin connection components of this metric in terms of  $W^2$ , the constraint Eq. (1.19) becomes

$$(d\chi)_{31} = \partial_2 H + H \partial_2 \log W^2, \quad (1.34)$$

which is the third condition in Eqs. (1.2), proving the first part of the theorem.

Showing that the inverse is also true, that is, that any metric of the form Eq. (1.1) satisfying the constraints Eqs. (1.2) is Kähler, is straightforward. One can introduce a Dreibein  $v^i$  given by Eqs. (1.23) and (1.32), a Vierbein as in Eqs. (1.6) and a complex structure as in Eq. (1.14). Equations (1.11) and (1.12) are automatically satisfied, and using the constraints (1.2) it is easy to verify that Eq. (1.13) which is again equivalent to Eq. (1.15), is also satisfied.

*Q.E.D.*

In the preceding discussion we have ignored the existence of a Kähler potential. Finding the Kähler potential from the metric in a given set of real coordinates is not an easy task. Observe, however, that the main equation that the functions that define our metric  $H, W$  satisfy, Eq. (1.3), can always be solved by introducing a real function  $\mathcal{K}(x^1, x^2, x^3)$  and defining

$$H \equiv \partial_2^2 \mathcal{K}, \quad W^2 \equiv -H^{-1} \left( \partial_1^2 + \partial_3^2 \right) \mathcal{K}. \quad (1.35)$$

The components  $\chi_1, \chi_3$  of the 1-form  $\chi$  satisfying Eq. (1.2) can also be derived from  $\mathcal{K}$ , as long as we choose coordinates such that  $\chi_2 = 0$ . They are given by

$$\chi_1 = -\partial_3 \partial_2 \mathcal{K}, \quad \chi_3 = \partial_2 \partial_1 \mathcal{K}. \quad (1.36)$$

It is tempting to identify  $\mathcal{K}$  with the Kähler potential. However, although this is likely to be the case, we have not proven its existence nor we have proven that the above relations are the unique way of solving the equations that define the metric. Nevertheless, we can always consider metrics constructed in this way since they are automatically Kähler metrics with a holomorphic isometry.

## 2 Special cases

The scalar curvature of the metric (1.1) can be written in the compact form

$$\hat{R} = \hat{\nabla}^2 \log W^2 = H^{-1} \bar{\nabla}^2 \log W^2, \quad (2.1)$$

where  $\hat{\nabla}^2$  is the 4-dimensional Laplacian operator. If one were to impose the requirement of scalar-flatness on the metric, this would thus translate to an equation for  $W^2$  which is known in the physics literature as the  $SU(\infty)$  or 3D Toda equation:

$$\left( \partial_1^2 + \partial_2^2 \right) \nu + \partial_2^2 e^\nu = 0, \quad (2.2)$$

with  $\nu \equiv \log W^2$ . In this case our result reduces to the one of LeBrun [10], which however was obtained imposing from the beginning a vanishing scalar curvature.

It is always possible to introduce two additional complex structures  $J^{(2,3)}$  satisfying together with  $J^{(1)} \equiv J$  the unit quaternionic algebra

$$J^{(x)} J^{(y)} = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} J^{(z)}. \quad (2.3)$$

In particular one can choose them to be of the form

$$J^{(2)} = \begin{pmatrix} i\sigma_2 & 0_{2 \times 2} \\ 0_{2 \times 2} & -i\sigma_2 \end{pmatrix}, \quad J^{(3)} = \begin{pmatrix} 0_{2 \times 2} & -i\sigma_2 \\ -i\sigma_2 & 0_{2 \times 2} \end{pmatrix}. \quad (2.4)$$

Observe that, with this choice, the 1-forms  $v^1$  and  $v^3$  can be written in terms of these complex structures in a similar way as  $v^2$  in (1.25), namely

$$v^1 = \iota_k J^{(2)}, \quad v^3 = -\iota_k J^{(3)}. \quad (2.5)$$

Of course in general these complex structures are not covariantly constant, in fact one has

$$\hat{\nabla}_m J^{(2)}_{np} = \hat{P}_m J^{(3)}_{np}, \quad (2.6)$$

$$\hat{\nabla}_m J^{(3)}_{np} = -\hat{P}_m J^{(2)}_{np}, \quad (2.7)$$

with the components of the 1-form  $P$  in (4-dimensional) flat indices given by



$$\hat{P}_m = \hat{J}_m^n \partial_n \log W. \quad (2.8)$$

Actually, the most general possible form for  $J^{(2,3)}$  would be

$$J^{(2)'} = \cos \theta J^{(2)} + \sin \theta J^{(3)}, \quad J^{(3)'} = \cos \theta J^{(3)} - \sin \theta J^{(2)}, \quad (2.9)$$

for some function  $\theta$ , in which case

$$\hat{P}' = \hat{P} - d\theta. \quad (2.10)$$

If one chooses  $H = \partial_{\underline{2}} \log W^2$ , then the integrability condition (1.3) reduces to the derivative with respect to  $x^2$  of the Toda equation (2.2). Therefore it is automatically satisfied if one imposes Eq. (2.2), which as we have seen is equivalent to the requirement of scalar-flatness. In this case one gets

$$\chi = \partial_{\underline{1}} \log W^2 dx^3 - \partial_{\underline{3}} \log W^2 dx^1 \quad (2.11)$$

and

$$\hat{P} = \frac{1}{2} dz, \quad (2.12)$$

which means that the complex structures given by (2.9) with  $\theta = z/2$  are covariantly constant and the space is hyperKähler, while not being preserved by the isometry. These hyperKähler metrics with a mono-holomorphic isometry were studied in [7, 8, 9].

If instead  $W$  is taken to be constant, the 3-dimensional metric is flat and the constraint Eqs. (1.2) reduce to

$$d\chi = \star_3 dH, \quad (2.13)$$

which implies that  $H$  is harmonic. The 1-form  $\hat{P}$  vanishes, which means that  $J^{(2,3)}$  are covariantly constant. In this case they are also preserved by the isometry,  $\mathcal{L}_k J^{(2,3)} = 0$ . The metric Eq. (1.1) is, then, a Gibbons-Hawking metric [5, 6]. Therefore, it is a hyper-Kähler metric admitting a triholomorphic isometry. In this scheme, the non-triviality of the conformal factor  $W$  can be seen as the obstruction for the Kähler metric with a holomorphic isometry to be a hyper-Kähler metric with a triholomorphic isometry.

### 3 An example

A non-trivial example of Kähler manifold admitting one isometry is the non-compact symmetric space  $\overline{\mathbb{CP}}^2 = \text{SU}(1,2)/\text{U}(2)$ . In supergravity it arises as the base space of  $\text{AdS}_5$ , which can be constructed as a  $\text{U}(1)$  bundle over  $\overline{\mathbb{CP}}^2$  [12]<sup>1</sup>. Its metric is usually given in terms of complex coordinates  $\zeta^i$ ,  $i = 1, 2$ , as

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<sup>1</sup>This is the non-compact version of the Hopf fibrations studied by Trautman in Ref. [13].

$$\mathcal{G}_{ij^*} = \frac{\delta_{ij^*}}{1 - \zeta^k \zeta^{*k^*}} + \frac{\zeta^{*i^*} \zeta^j}{(1 - \zeta^k \zeta^{*k^*})^2}. \quad (3.1)$$

Introducing the real coordinates

$$\zeta^1 = \tanh \rho \cos \frac{\theta}{2} e^{-\frac{i}{2}(\psi + \varphi)}, \quad \zeta^2 = \tanh \rho \sin \frac{\theta}{2} e^{-\frac{i}{2}(\psi - \varphi)}, \quad (3.2)$$

the line element of  $\overline{\mathbb{CP}}^2$  takes the form

$$ds^2 = d\rho^2 + \frac{1}{4} \sinh^2 \rho \left[ d\theta^2 + \sin^2 \theta d\varphi^2 + \cosh^2 \rho (d\psi + \cos \theta d\varphi)^2 \right], \quad (3.3)$$

and with the further coordinate change

$$z = \psi, \quad x^2 = \frac{1}{4} \sinh^2 \rho, \quad x^1 = \tan \frac{\theta}{2} \cos \varphi, \quad x^3 = \tan \frac{\theta}{2} \sin \varphi, \quad (3.4)$$

it can be brought to the form (1.1), with the functions  $H$ ,  $W^2$  and 1-form  $\chi$  that define it given by

$$\begin{aligned} H^{-1} &= x^2(1 + 4x^2), \\ W^2 &= \frac{4x^2}{H[1 + (x^1)^2 + (x^3)^2]^2}, \\ \chi &= \frac{[1 - (x^1)^2 - (x^3)^2] x^1 dx^3 - x^3 dx^1}{[1 + (x^1)^2 + (x^3)^2] (x^1)^2 + (x^3)^2}. \end{aligned} \quad (3.5)$$

The functions  $W$  and  $H$  for this metric have been given in Ref.

## 4 Conclusions

With the result we have just proven, the conditions that determine the fields of supersymmetric solutions of FI-U(1)-gauged minimal supergravity in 5 and 6 dimensions must become a set of partial differential equations on a set of real functions, just as in the ungauged case, although here we expect the equations to be coupled and non-linear. Still making use of the Ansatz in Eq. (1.1), (1.2) should simplify considerably the problem. Work in this direction is in progress [16, 17].

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